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Abstract
The thermodynamic limit of the Lipkin model is investigated. While the limit
turns out to be rather elusive, the analysis gives strong indications that the
limit yields two analytically dissociated operators, one for the normal and one
for the deformed phase. While the Lipkin Hamiltonian is Hermitian and has
a second-order phase transition in finite dimensions (finite particle number),
both properties seem to be destroyed in the thermodynamic limit.

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Introduction

The Lipkin model [1], even though it appeared more than 40 years ago in the literature,
has, up to present, given rise to a large number of research papers. This continuing interest
is due to its basic properties in that it demonstrates the mechanism of a phase transition
including symmetry breaking for a schematic many-body system. Originally introduced as
a model in nuclear physics [2], it has found applications in a broad range of other topics:
statistical mechanics of quantum spin systems [3], Bose–Einstein condensates [4] as well as
quantum entanglement [5], to name but a few. Recent interest is focused upon its large N
behaviour with N being the particle number. Progress has been achieved using a variety of
methods such as continuous unitary transformations [6, 7], a semiclassical approach [8] or the
Bethe ansatz [9].

   We briefly list here the major results relevant to the present paper. In its original form, the
Lipkin model considers interacting fermions occupying two Ω-fold degenerate levels. One
major appeal is due to its easy solubility. This is achieved by rewriting the kinetic energy term
of the fermion Hamiltonian, namely,

$$\sum_{k,\sigma} \sigma a^{\dagger}_{k,\sigma} a_{k,\sigma}$$

and the two-body interaction term

$$\sum_{k,k',\sigma} a^{\dagger}_{k,\sigma} a^{\dagger}_{k',\sigma} a_{k',-\sigma} a_{k,-\sigma}$$

in terms of the $SU(2)$ operators J_z and $J_{\pm}^2$, respectively. In dimensionless form, the Hamiltonian thus reduces to

$$H(\lambda) = J_z + \frac{\lambda}{2N}\left(J_+^2 + J_-^2\right), \tag{1}$$

where the $(2j + 1 = N + 1)$-dimensional representations of the $SU(2)$ operators are used with $N \leqslant \Omega$ being the particle number. Here, the interaction is scaled by N to ensure that H is extensive, the operators J_+^2 and J_-^2 effectively scale as N^2. The Hamiltonian allows reduction into two spaces: m integer and m half-integer, with m the eigenvalues of J_z; it corresponds to N-even and N-odd, respectively, and is denoted as parity. For $\lambda \gtrsim 0$, the N-even and N-odd levels are obviously separated and remain so for all $\lambda < 1$, while the levels become degenerate for $\lambda > 1$. The phase at $\lambda < 1$ is called the normal phase while the symmetry (parity) breaking phase at $\lambda > 1$ is called the deformed phase. For $\lambda \gtrsim 1$, this phase transition is confined to the lower part of the spectrum in that higher up in the spectrum the normal phase still prevails. In fact, for a specific $\lambda > 1$, there is an energy $E_{k_c}(\lambda)$ such that the ordered spectrum E_k is, for $k < k_c$, associated with the deformed phase while for $k > k_c$, the energies relate to the normal phase [10]. In other words, the transition moves up in the spectrum from the ground state E_1 to $E_{N/2}$ when λ increases from unity to infinity (we confine ourselves to the lower half of the spectrum as the symmetry of the model yields the same pattern in the upper half but mirror reflected). For finite N, E_{k_c} is clearly characterized by the minimum of the level distance; this minimum gap vanishes for $N \to \infty$. In fact, at $\lambda = 1$, the large N scaling behaviour [7, 8] is found,

$$\Delta E_k \sim \left(\frac{k}{N}\right)^{1/3}, \tag{2}$$

whereas for $\lambda > 1$, the gap around E_{k_c} behaves as [8]

$$\Delta E = \frac{2\pi\sqrt{\lambda^2 - 1}}{\ln N}. \tag{3}$$

We mention that a mean field approach yields the result [11]

$$E_k(\lambda) = k\sqrt{1 - \lambda^2} \qquad \text{for } \lambda < 1 \tag{4}$$

$$E_k(\lambda) = k\sqrt{2(\lambda^2 - 1)} \qquad \text{for } \lambda > 1 \tag{5}$$

being obviously unable to reproduce the richer structure described above; however, the last two equations do describe the lower part of the spectrum remarkably well.

In the present paper, we aim at a more complete description of the thermodynamic limit of the Lipkin model.

The spectrum

A closer look at the essential features of the spectrum for large N may elucidate statements made in the previous section and provides clues of what can be expected in the limit $N \to \infty$. In figure 1, the spectra are drawn for a few values of λ. As this is done for large values of N, we use the 'continuous' variable $x = 2k/N$ ranging from zero to unity and plot $\epsilon(x) = 2E_k/N$ versus x. Using this scale, we note the following features: for $\lambda \geqslant 1$ each spectrum has a singularity when it crosses the critical line $\epsilon = -1$. In fact, the singularity is a point of inflection with a zero derivative and an infinite second derivative as shown in figure 2 and in accordance with equation (3) (for $\lambda = 1$, the spectrum just touches the critical line with zero
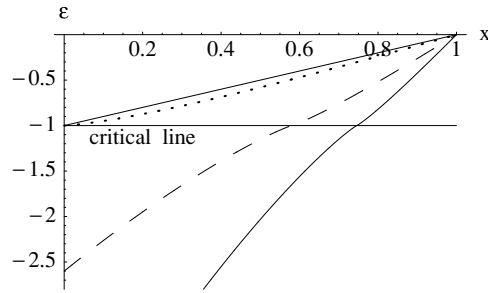
Figure 1. Spectra for $\lambda = 0$ (solid line), $\lambda = 1$ (dotted line), $\lambda = 5$ (dashed line) and $\lambda = 10$ (solid line).
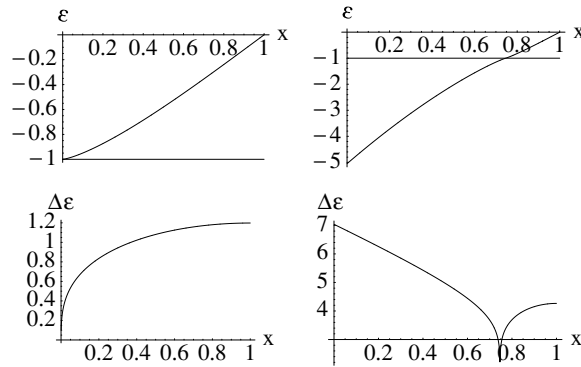
Figure 2. Spectra (top row) and their derivatives (bottom row) for $\lambda = 1$ (left column) and $\lambda = 10$ (right column). The top row is displayed for convenience; it shows the same respective curves including the critical line as those in figure 1.

first and infinite second derivative). In other words, the level density is singular at the critical line. Note that the singularity occurring at the critical line is common to all spectra for $\lambda > 1$ irrespective of the particular value for λ. For $\lambda < 1$ the spectra, i.e. their level densities, are smooth.

The range $0 > \epsilon > -1$ is the normal phase while $\epsilon < -1$ corresponds to the deformed phase. For better illustration, we do not display the lower end of figure 1; it suffices to note that [8] for $\lambda > 1$ the ground state assumes the value

$$\epsilon(0) = -\frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right). \tag{6}$$

The critical line is crossed by each level sequence at a certain $x_c(\lambda)$ associated with $\lambda > 1$; the larger λ the larger x_c, i.e. the phase transition occurs higher up in the spectrum and higher above the ground state given in equation (6). The critical line is special not only as it signals the phase transition through the singularity, but it is also associated with a specific state vector being identical to the unperturbed ground state once the limit $N \to \infty$ is attained. In fact, it is

$$\frac{2}{N}H(\lambda)|j, -j\rangle = -|j, -j\rangle + \sqrt{2}\frac{\lambda}{N}|j, -j + 2\rangle, \tag{7}$$
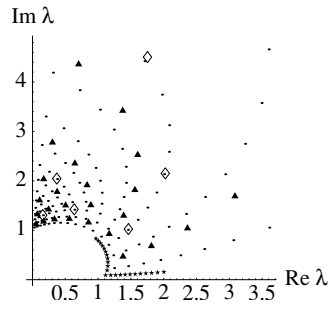
Figure 3. Exceptional points in the λ-plane for $N = 8$ (diamonds), $N = 16$ (triangles) and $N = 32$ (dots). The first 12 points along the real axis and the innermost arc running from the real to the imaginary axis are indicated by asterisks for $N = 96$.

where the second term of the right-hand side vanishes in the limit. The right-hand side also becomes independent of λ when $N \to \infty$. Recall that for finite N, all levels and state vectors are analytic functions of λ and analytically connected in the λ-plane. As a special result, we further note that this state is perfectly localized; all other states are more or less extended superpositions of vectors $|j, m\rangle, -j \leqslant m \leqslant j$; this localization was also noted in [8].

The singularity discussed above for the level density is found as well when a particular level (fixed value of x) is plotted versus λ. The same pattern prevails: there is a point of inflection at $\lambda = \lambda_c$ (associated with $\epsilon(x_c) = -1$) with a vanishing derivative and an infinite second derivative in λ.

Exceptional points

The whole pattern, in particular the singular behaviour at the critical line, i.e. at the phase transition, can be understood from the global analytical structure of the spectrum as a function of the complex parameter λ. The exceptional points (EP) [12] are the only singularities of the spectrum for the underlying model. For a Hermitian problem they can occur only in the complex λ-plane. They are square root branch points where, generically, two energies are connected by a branch point. For the Hamiltonian (1) all energies $E_k(\lambda)$ are analytically connected; in fact, each $E_k(\lambda)$ represents the values on a particular Riemann sheet of one analytic function [13].

In figure 3, we illustrate the exceptional points for a few small values of N. Two aspects are of importance: (i) for increasing N some EPs accumulate along the real axis $\lambda \geqslant 1$ and, (ii) at the EPs of the sequence along the real axis, starting at $\lambda = 1$, the coalescence at the first EP affects the two levels E_1 and E_2, at the second EP the levels E_2 and E_3, at the third EP E_3 and E_4 and so forth.

The second point clarifies immediately some observations made in the previous section. Take, for instance, $\lambda = 2$. From figure 3, we read off that there are $N/8$ EPs along the real axis between $\lambda = 1$ and $\lambda = 2$. This implies that the Riemann sheet (energy level) with $k = N/8$, that is for $x = 0.25$, has a singularity at $\lambda = 2$. In general terms, there is just one x_c associated with a particular value of λ.

With regard to the first point we note two aspects that are, however, related. In the limit $N \to \infty$ there is a dense set of EPs, say at $\lambda = 2$, for $x = 0.25$. The singularity, being an accumulation of square root branch points, cannot be expected to be of a square root nature. The more likely behaviour is that of a logarithmic singularity. This is supported by a

similar, albeit simpler situation [14], where an accumulation of square root branch points has been demonstrated analytically to lead to a logarithmic branch point. There is further strong numerical evidence: a perfect fit is obtained of the curves in figures 1 and 2 using the trial functions

$$(x - x_c)^2 (a_1 \ln|x - x_c| + a_2 (\ln|x - x_c|)^2 + \cdots) \tag{8}$$

employing three terms or less. The trial function (8) is used to fit the spectra of figure 1 (after a shift upward by one unit to make the critical line occurring at zero). In a subsequent acid test, the derivative of (8) is compared with the respective curves for $\Delta \epsilon$ in figure 2 yielding an excellent agreement. Note that the modulus of $x - x_c$ is being used implying that the function is non-analytic at this point. Depending on the sign of $x - x_c$, a separate fit must be made. As was mentioned in the previous section, the same pattern is found when a level is considered as a function of λ. A good fit is obtained replacing x and x_c by λ and λ_c, respectively, in (8).

Not only do we have evidence for a very special singularity, the fact that the EPs accumulate on the real axis indicates that the thermodynamic limit of the Hamiltonian (1) *cannot be a Hermitian operator*. Moreover, if the limit is taken either in the deformed or in the normal regime, the two regimes become disconnected for real values of λ.

Conclusion

We summarize our findings: the thermodynamic limit $N \to \infty$ of the Hamiltonian (1) leads to a specific singular behaviour associated with the phase transition. It is immaterial as to whether a specific level is considered as a function of λ or whether the whole spectrum is considered for a fixed value of λ. In either case, a logarithmic singularity is found at a particular value of λ being uniquely related to a specific level. Note that the singularity is non-analytic in that, on the real λ-axis, the behaviour on the left-hand side of the singularity is not the continuation of that on the right-hand side. The singularity produces an infinity in the respective second derivative. As such, it is no longer a second-order phase transition as it is found for finite N.

Furthermore, the fact that the EPs accumulate on the real λ-axis in the limit renders the operator non-Hermitian in the limit. In fact, depending on how the limit is taken, a different result emerges: one operator describing the normal phase and the other describing the deformed phase. In contrast to the case of finite N, the two are dissociated. This is in contrast to finite N, where all energies and state vectors are analytically connected. In particular, when the limit $N \to \infty$ is attained, the state vector $|j, -j\rangle$ (see equation (7)) is common to all spectra, irrespective of λ. As a particular feature, this vector has, for $\lambda > 0$, a non-zero overlap with any other vector of either the normal or the deformed sector. This is in striking difference to finite N, where (1) yields the usual complete orthonormal basis for any given λ.

It is worth to mention that the semiclassical approach [8] has explicitly given rise to ambiguities of a Hermitian thermodynamic limit. An unambiguous limit could be obtained only to lowest order in $\hbar \sim 1/N$.

We refrain from speculating about physical consequences of our findings. Rather we feel that the thermodynamic limit of the Lipkin model is unsuitable to describe a physical system despite its beauty for finite yet large values of N. We see the value of this analysis, apart from its own sake, in a specific contribution of the general understanding of models describing quantum phase transitions.

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